

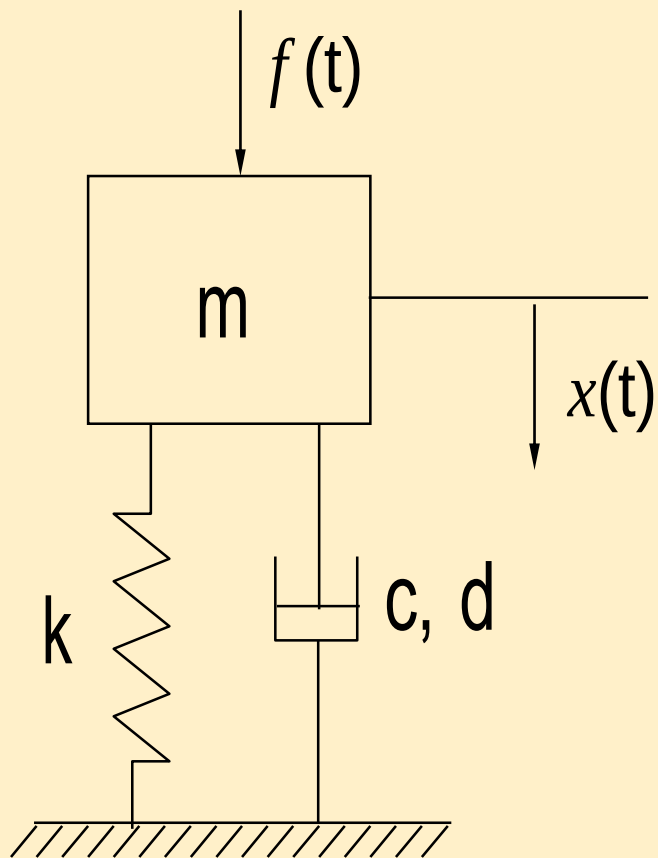


Mechanical Vibrations



Single DoF free vibration system

Single degree of freedom system with viscous damping:



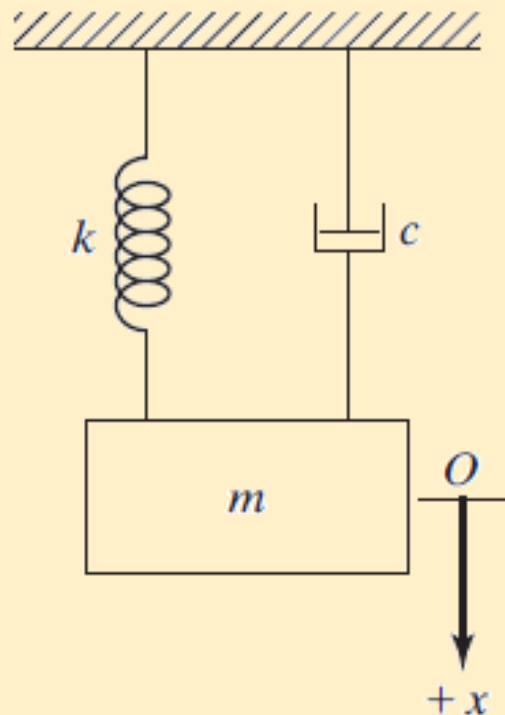
1- Hysteresis Damping (Structural)

2-Viscously-Damped

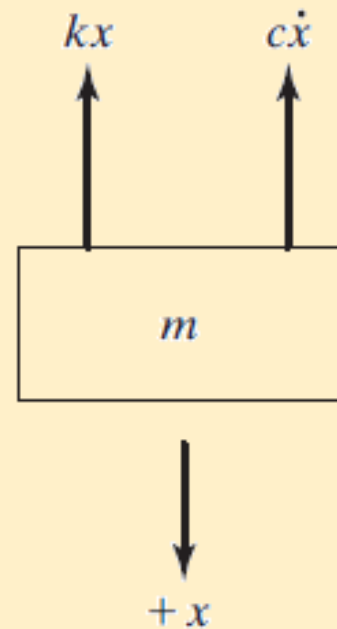
3- Friction Damped

Single DoF free vibration system

Single degree of freedom system with viscous damping:



System



Free-body diagram

Single DoF free vibration system

Mathematical model (governing equation of motion):

$$m\ddot{x} + c\dot{x} + kx = 0$$

$c\dot{x} = F_d$ is the damping force.

It is proportional to velocity

Solution:

Assume $x(t) = B e^{st}$, substitute into the equation of motion:

$$(ms^2 + cs + k)Be^{st} = 0 \Rightarrow s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$



Single DoF free vibration system

$$x_1(t) = C_1 e^{s_1 t} \quad \text{and} \quad x_2(t) = C_2 e^{s_2 t}$$

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

$$= C_1 e^{\left\{-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t} + C_2 e^{\left\{-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t}$$

Critical Damping Constant and the Damping Ratio.

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$



$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n$$

Single DoF free vibration system

damping ratio : $\zeta = \frac{C}{C_c} \longrightarrow \frac{C}{2m} = \zeta \omega_n$

$$\longrightarrow s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1} \right) \omega_n$$

In the face of the value of the square root shown above three possible solutions are exist. These solutions are as follows:



Single DoF free vibration system

Solution # 1: under-damped vibration:

When $\zeta < 1$

The roots $s_{1,2}$ of the characteristic equation can now be written as:

$$s_{1,2} = \left(-\zeta \pm i\sqrt{1-\zeta^2} \right) \omega_n$$

The solution for this case becomes :

$$x(t) = C_1 e^{(-\zeta + i\sqrt{1-\zeta^2})\omega_n t} + C_2 e^{(-\zeta - i\sqrt{1-\zeta^2})\omega_n t}$$

$$= e^{-\zeta\omega_n t} \left\{ C_1 e^{i\sqrt{1-\zeta^2}\omega_n t} + C_2 e^{-i\sqrt{1-\zeta^2}\omega_n t} \right\}$$

Single DoF free vibration system

$$= e^{-\zeta\omega_n t} \left\{ (C_1 + C_2) \cos \sqrt{1 - \zeta^2} \omega_n t + i(C_1 - C_2) \sin \sqrt{1 - \zeta^2} \omega_n t \right\}$$

Real Initial Condition:

$$= e^{-\zeta\omega_n t} \left\{ C'_1 \cos \sqrt{1 - \zeta^2} \omega_n t + C'_2 \sin \sqrt{1 - \zeta^2} \omega_n t \right\}$$

$$= X_0 e^{-\zeta\omega_n t} \sin \left(\sqrt{1 - \zeta^2} \omega_n t + \phi_0 \right)$$

Or:

$$= X e^{-\zeta\omega_n t} \cos \left(\sqrt{1 - \zeta^2} \omega_n t - \phi \right)$$

Single DoF free vibration system

For the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$

$$C'_1 = x_0 \quad \text{and} \quad C'_2 = \frac{\dot{x}_0 + \zeta \omega_n x_0}{\sqrt{1 - \zeta^2} \omega_n}$$

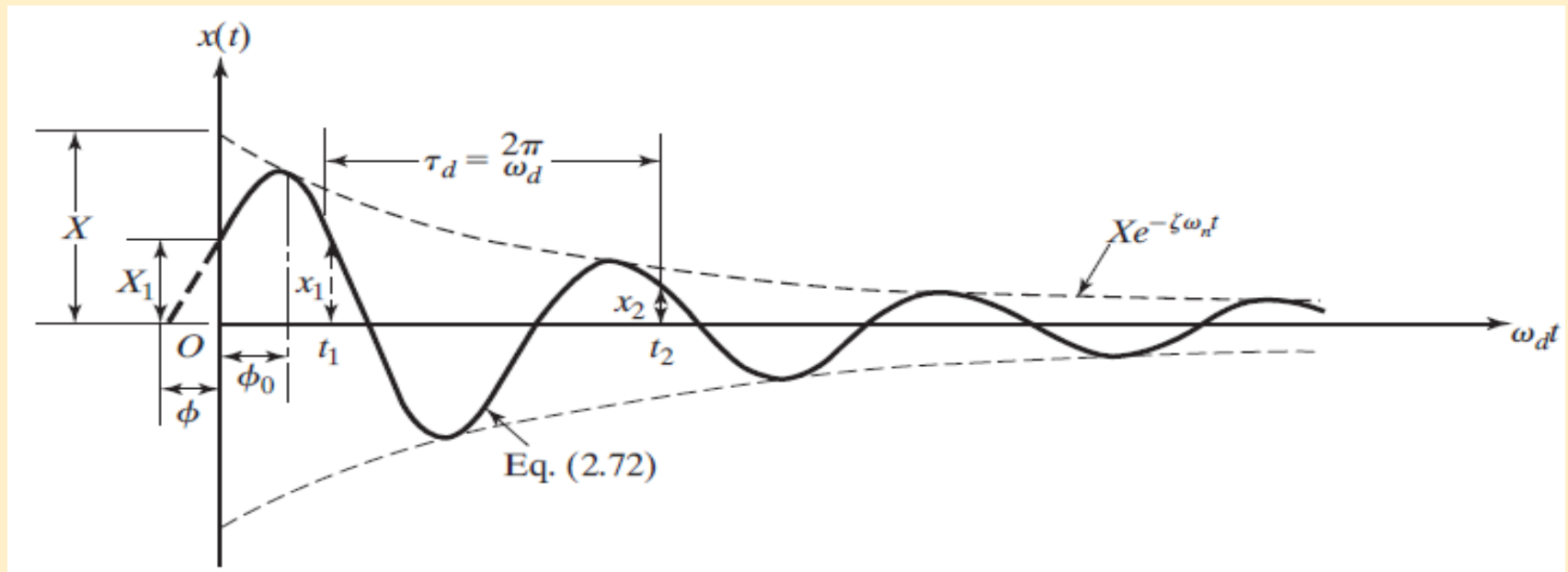
$$x(t) = e^{-\zeta \omega_n t} \left\{ x_0 \cos \sqrt{1 - \zeta^2} \omega_n t + \frac{\dot{x}_0 + \zeta \omega_n x_0}{\sqrt{1 - \zeta^2} \omega_n} \sin \sqrt{1 - \zeta^2} \omega_n t \right\}$$



Single DoF free vibration system

Frequency of damped vibration:

$$\omega_d = \sqrt{1 - \zeta^2} \omega_n$$



Under-damped vibration

Single DoF free vibration system

Solution # 2: critically damped vibration ($\zeta=1$):

For this case the roots of the characteristic equation become:

$$s_{1,2} = -\omega_n$$

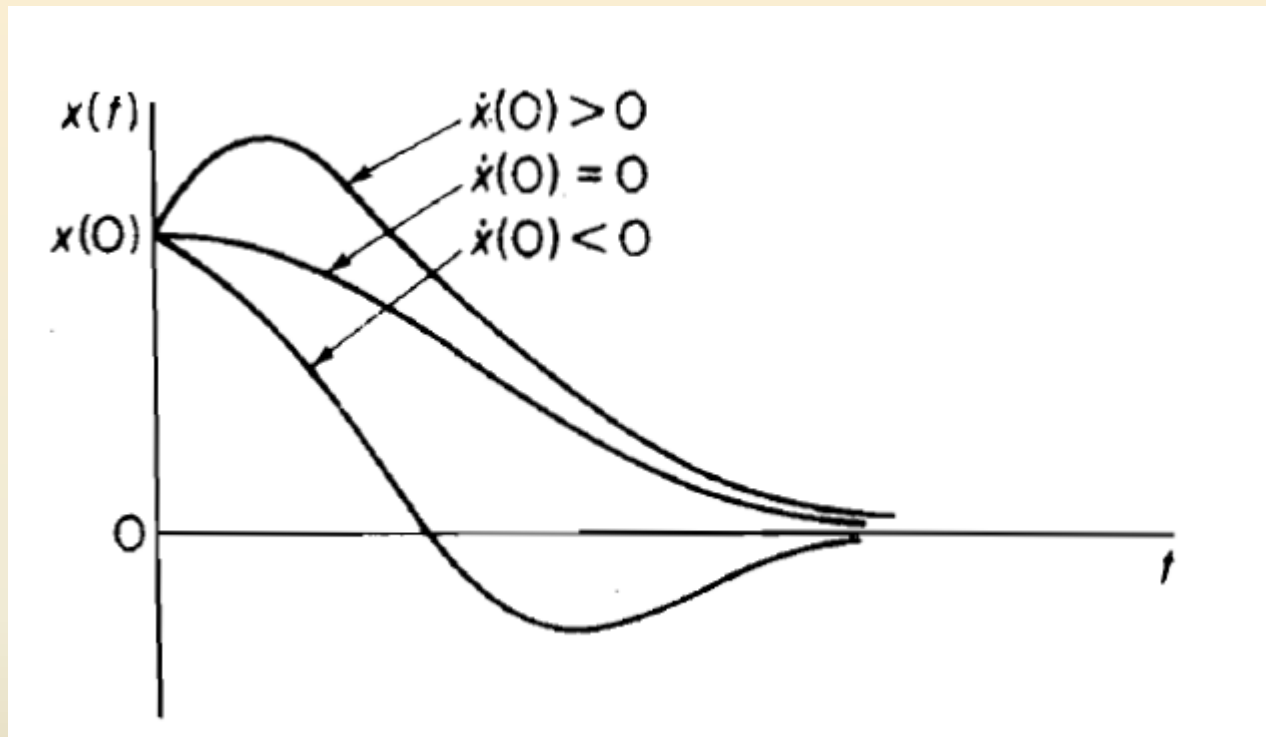
Therefor the solution can be written as :

$$x(t) = e^{-\omega_n t} \{B_1 + B_2 t\}$$

B_1 and B_2 are constants that can be determined from inatial conditions. The motion is no longer harmonic as shown in the figure.



Single DoF free vibration system



- The system returns to the equilibrium position in short time
- The shape of the curve depends on initial conditions as shown

Single DoF free vibration system

Solution # 3: over-damped vibration ($\zeta > 1$)

For this case the roots of the characteristic equation become:

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1} \right) \omega_n$$

Therefore, the solution can be written as,

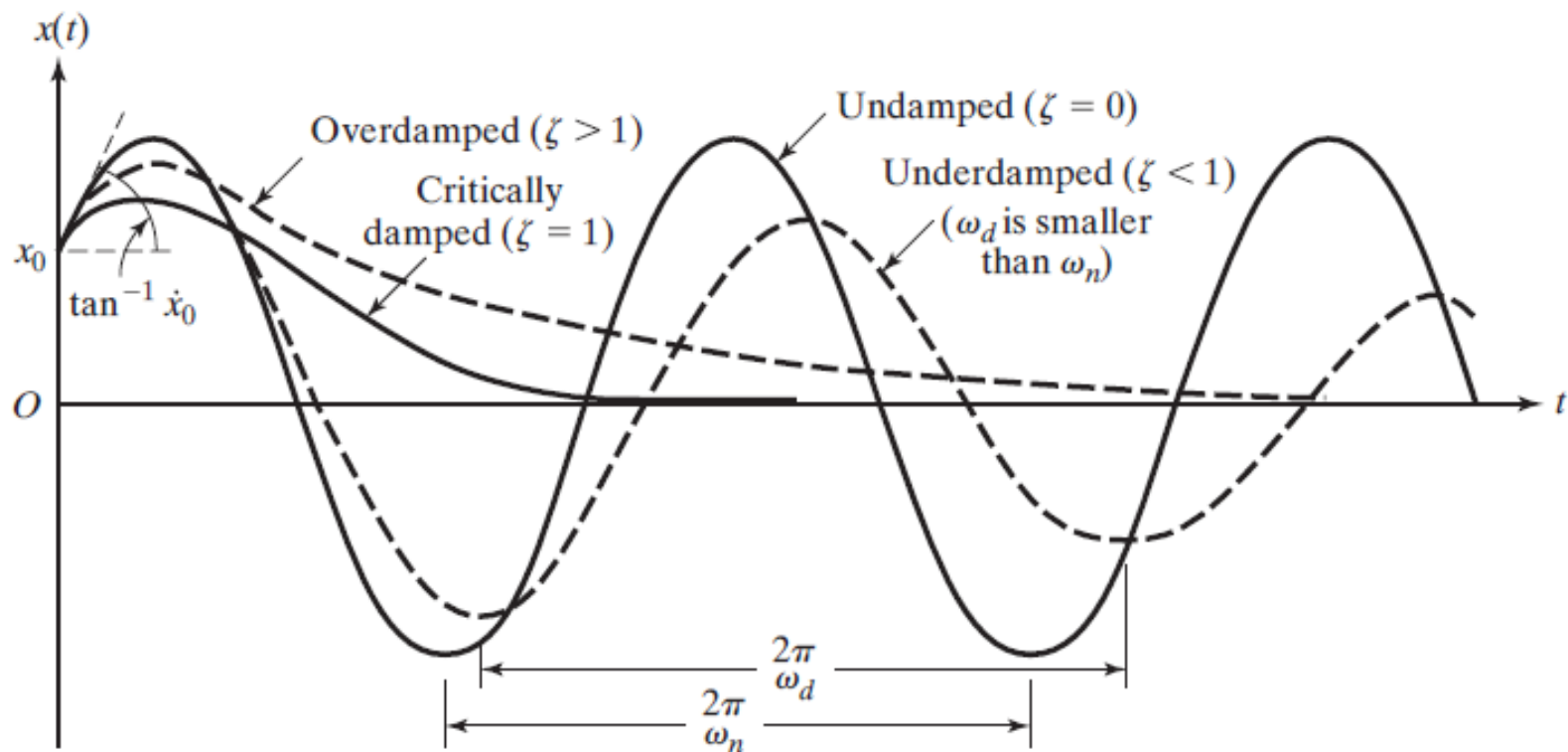
$$x(t) = B_1 e^{-\left(\zeta + \sqrt{\zeta^2 - 1} \right) \omega_n t} + B_2 e^{-\left(\zeta - \sqrt{\zeta^2 - 1} \right) \omega_n t}$$

B_1 and B_2 are constants to be determined from knowing the initial conditions of the motion.



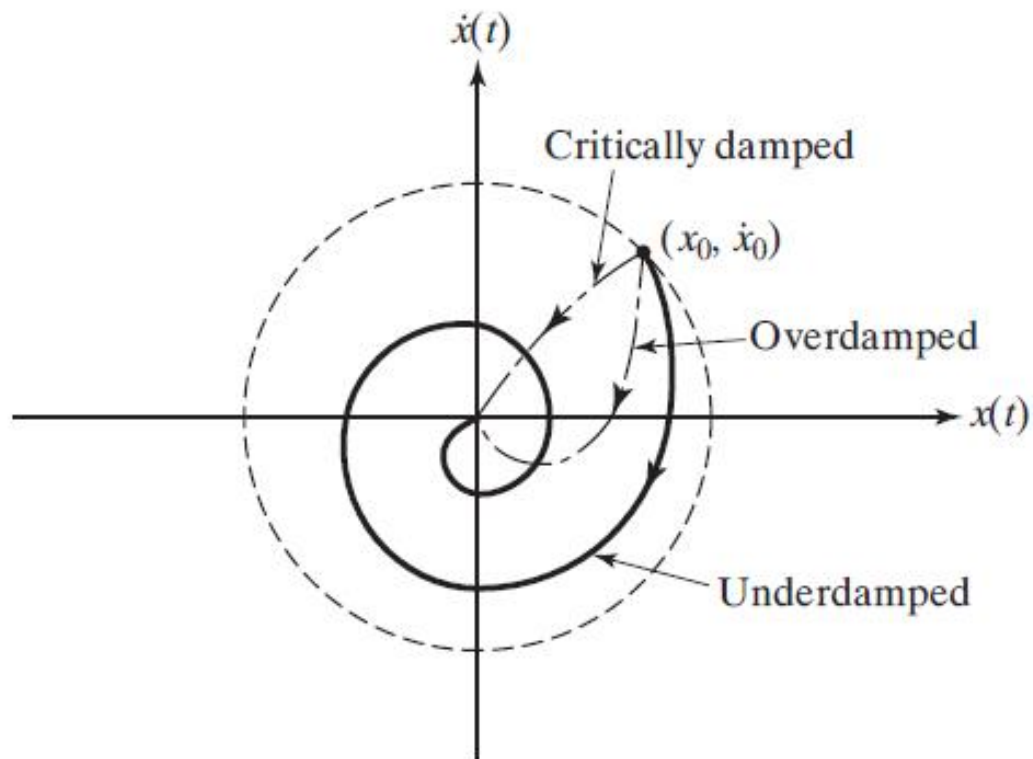
Single DoF free vibration system

Graphical representation of the motions of the damped systems



Single DoF free vibration system

Phase plane of a damped system.



Single DoF free vibration system

Logarithmic decrement

The logarithmic decrement represents the rate at which the amplitude of a free-damped vibration decreases. It is defined as the natural logarithm of the ratio of any two successive amplitudes.



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Logarithmic decrement:

$$\frac{x(t_1)}{x(t_2)} = \frac{Ae^{-\zeta\omega_n t_1} \cos(\omega_d t_1 - \phi)}{Ae^{-\zeta\omega_n t_2} \cos(\omega_d t_2 - \phi)}$$

But

$$t_2 = t_1 + \tau_d \Rightarrow \tau_d = \frac{2\pi}{\omega_d}$$
$$\cos(\omega_d t_2 - \phi) = \cos(2\pi + \omega_d t_1 - \phi)$$
$$= \cos(\omega_d t_1 - \phi)$$

So

$$\frac{x_1(t)}{x_2(t)} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1 + \tau_d)}} = e^{\zeta\omega_n \tau_d}$$

Assume: δ is the logarithmic decrement

$$\delta = \ln \frac{x_1(t)}{x_2(t)} = \zeta\omega_n \tau_d = \zeta\omega_n \frac{2\pi}{\sqrt{1-\zeta^2} \omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

Logarithmic decrement: is dimensionless

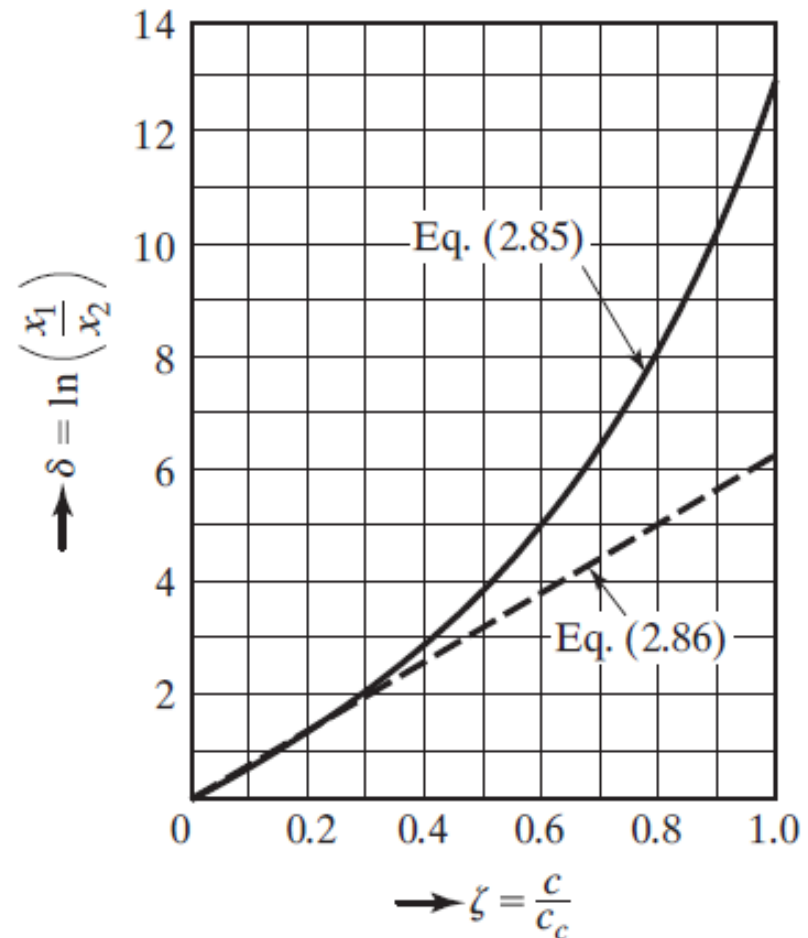
Single DoF free vibration system

Logarithmic decrement

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

For small damping ; $\zeta \ll 1$

$$\delta \approx 2\pi\zeta$$



Single DoF free vibration system

Generally, when the amplitude after a number of cycles “n” is known, logarithmic decrement can be obtained as follows:

$$\frac{X_1}{X_n} = \frac{X_1}{X_2} \times \frac{X_2}{X_3} \times \dots \times \frac{X_{n-1}}{X_n}$$

$$\text{Ln} \left(\frac{X_1}{X_n} \right) = \text{Ln} \left(\frac{X_1}{X_2} \times \frac{X_2}{X_3} \times \dots \times \frac{X_{n-1}}{X_n} \right)$$

$$= \text{Ln} \left(\frac{X_1}{X_2} \right) + \text{Ln} \left(\frac{X_2}{X_3} \right) + \dots + \text{Ln} \left(\frac{X_{n-1}}{X_n} \right)$$

$$= \delta + \delta + \dots + \delta = (n-1)\delta$$

$$\ln \left(\frac{X_1}{X_n} \right) = (n-1)\delta = (n-1) \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$



Single DoF free vibration system

Example 2.6

An under-damped shock absorber is to be designed for a motorcycle of mass 200 kg (Fig.(a)). When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as indicated in Fig.(b).

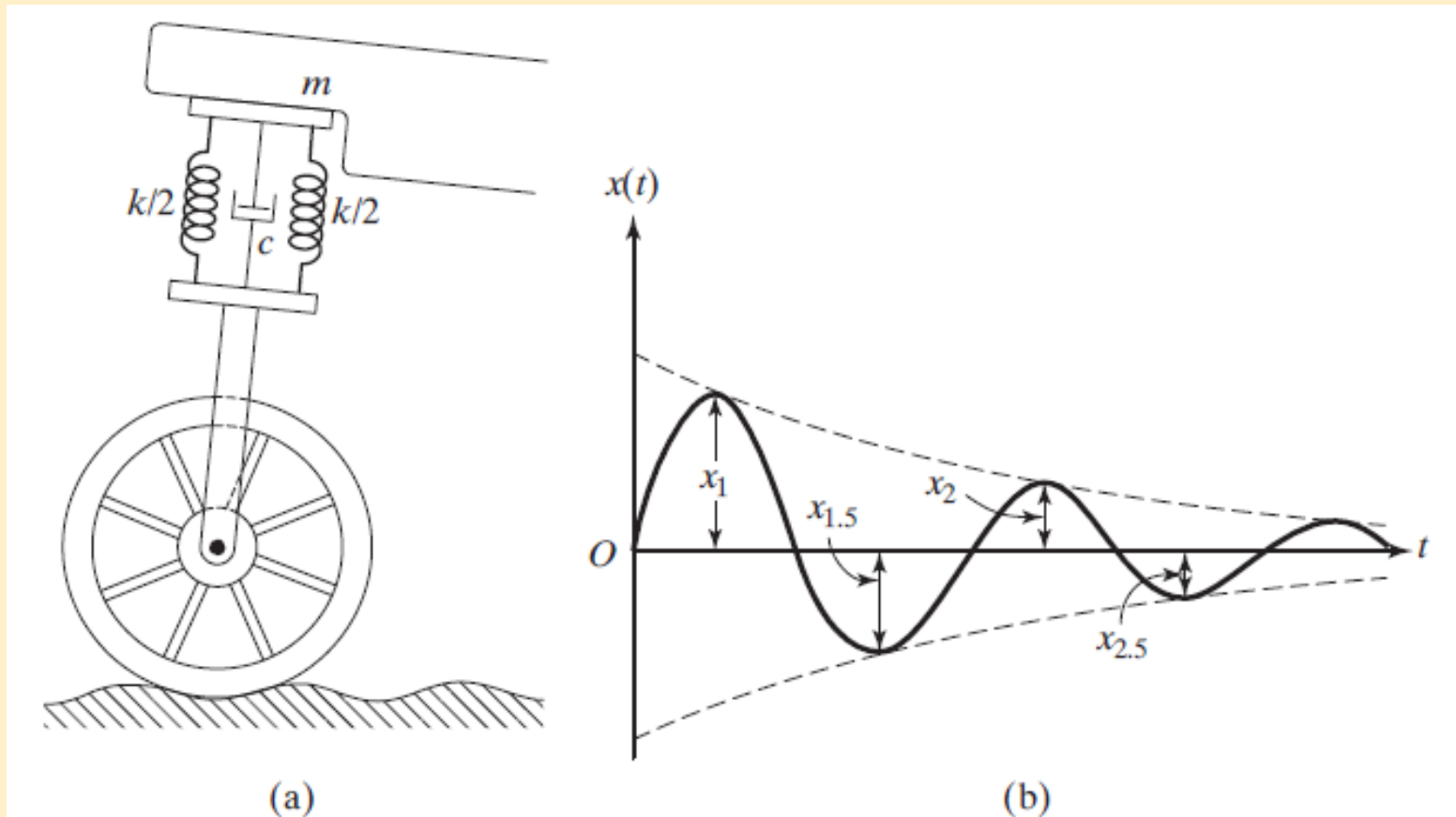
Requirements:

1. Find the necessary stiffness and damping constants of the shock absorber if the damped **period** of vibration is to be **2s** and the amplitude x_1 is to be reduced to one-fourth in one half cycle (i.e. $x_{1.5} = x_1/4$).

Single DoF free vibration system

Example 2.6

Note that this system is under-damped system



Single DoF free vibration system

Solution:
Finding k and c

Here, $n = 1.5$ and $\frac{x_1}{x_n} = 4$, then the logarithmic decrement is:

$$\delta = \frac{1}{n-1} \ln \frac{x_1}{x_n} = \frac{1}{0.5} \ln(4) = 2.7726 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\Rightarrow \zeta = 0.4037$$

$$\tau_d = 2 = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}} \Rightarrow \omega_n = \frac{\pi}{\sqrt{1-0.4037^2}} = 3.4338 \text{ rad / s}$$



Single DoF free vibration system

The critical damping can be found as:

$$c_c = 2m\omega_n = 2(200)(3.4338) = 1373.54 \text{ N.sec/m}$$

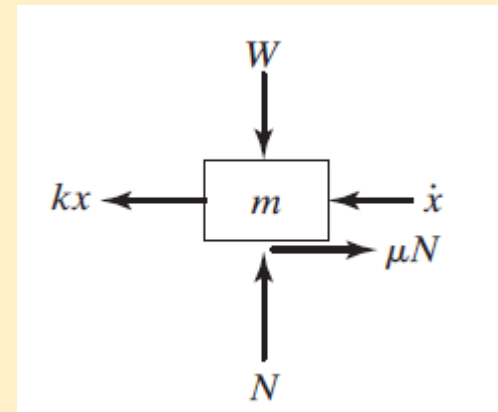
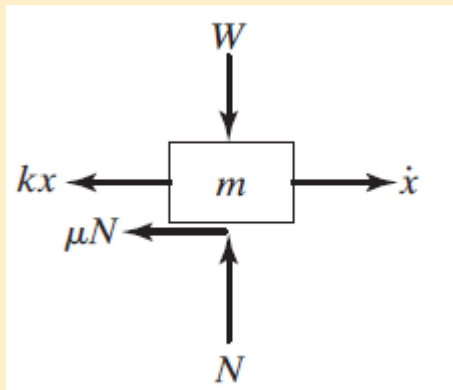
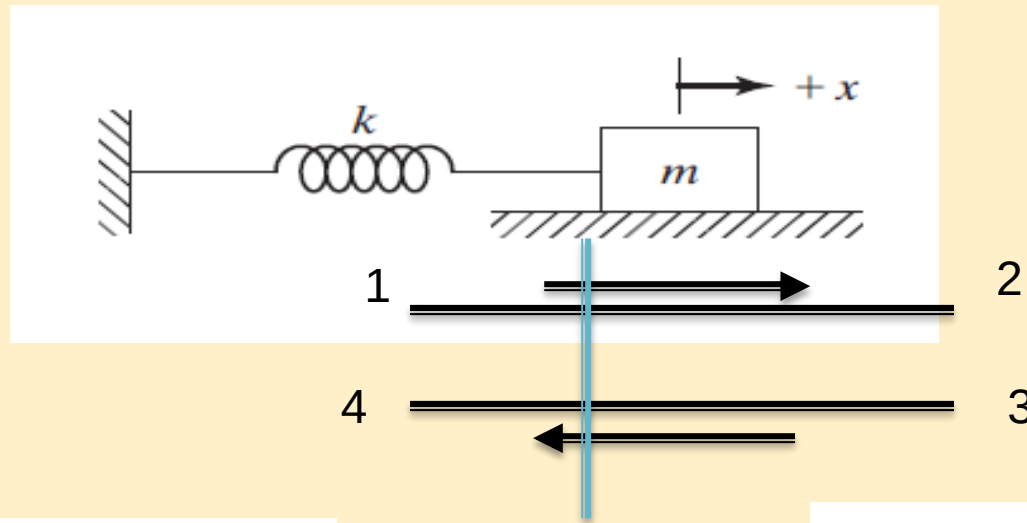
Damping coefficient C and stiffness K can be found as:

$$c = \zeta c_c = (0.4037)(1373.54) = 554.4981 \text{ N.sec/m.....(Ans.)}$$

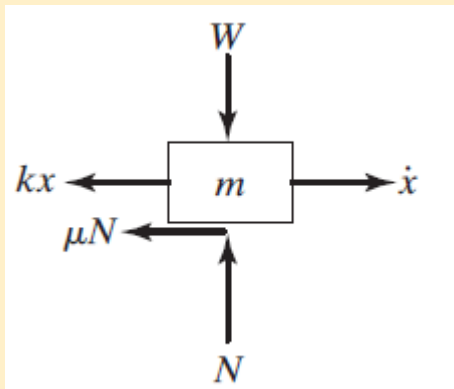
$$k = m\omega_n^2 = (200)(3.4338)^2 = 2358.2652 \text{ N/m.....(Ans.)}$$

Single DoF free vibration system

The Coulomb damping:



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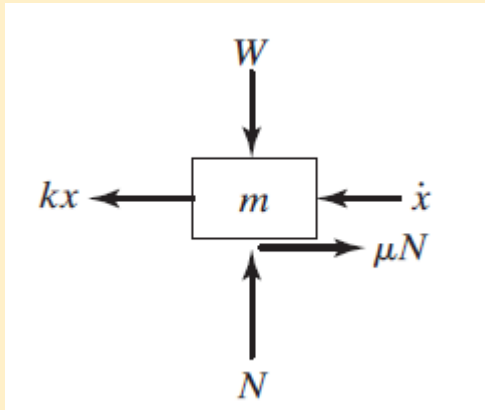


$$m\ddot{x} = -kx - \mu N$$



$$m\ddot{x} + kx = -\mu N$$

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t - \frac{\mu N}{k}$$



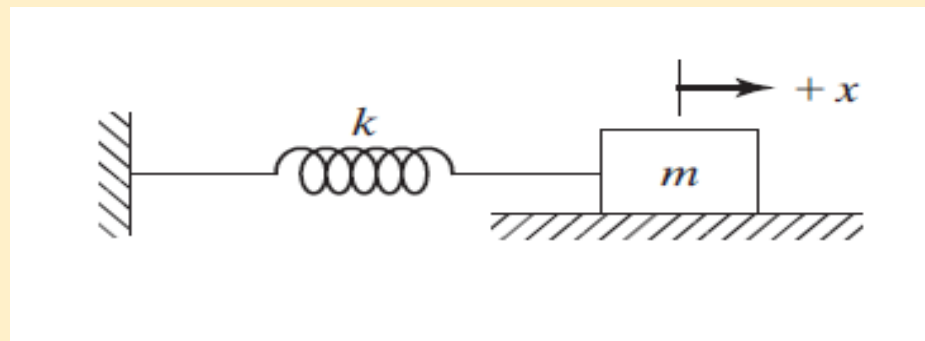
$$-kx + \mu N = m\ddot{x}$$



$$m\ddot{x} + kx = \mu N$$

$$x(t) = A_3 \cos \omega_n t + A_4 \sin \omega_n t + \frac{\mu N}{k}$$

Single DoF free vibration system



$$m\ddot{x} + \mu mg \operatorname{sgn}(\dot{x}) + kx = 0$$



Single DoF free vibration system

Sol.

$$x(t) = A_3 \cos \omega_n t + A_4 \sin \omega_n t + \frac{\mu N}{k}$$



$$x(t = 0) = x_0$$

$$\dot{x}(t = 0) = 0$$

$$A_3 = x_0 - \frac{\mu N}{k}, \quad A_4 = 0$$

$$x(t) = \left(x_0 - \frac{\mu N}{k} \right) \cos \omega_n t + \frac{\mu N}{k}$$

$$0 \leq t \leq \pi / \omega_n$$



Single DoF free vibration system

Sol.

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t - \frac{\mu N}{k}$$



$$x(t = 0) = x\left(t = \frac{\pi}{\omega_n}\right) = \left(x_0 - \frac{\mu N}{k}\right) \cos \pi + \frac{\mu N}{k} = -\left(x_0 - \frac{2\mu N}{k}\right)$$

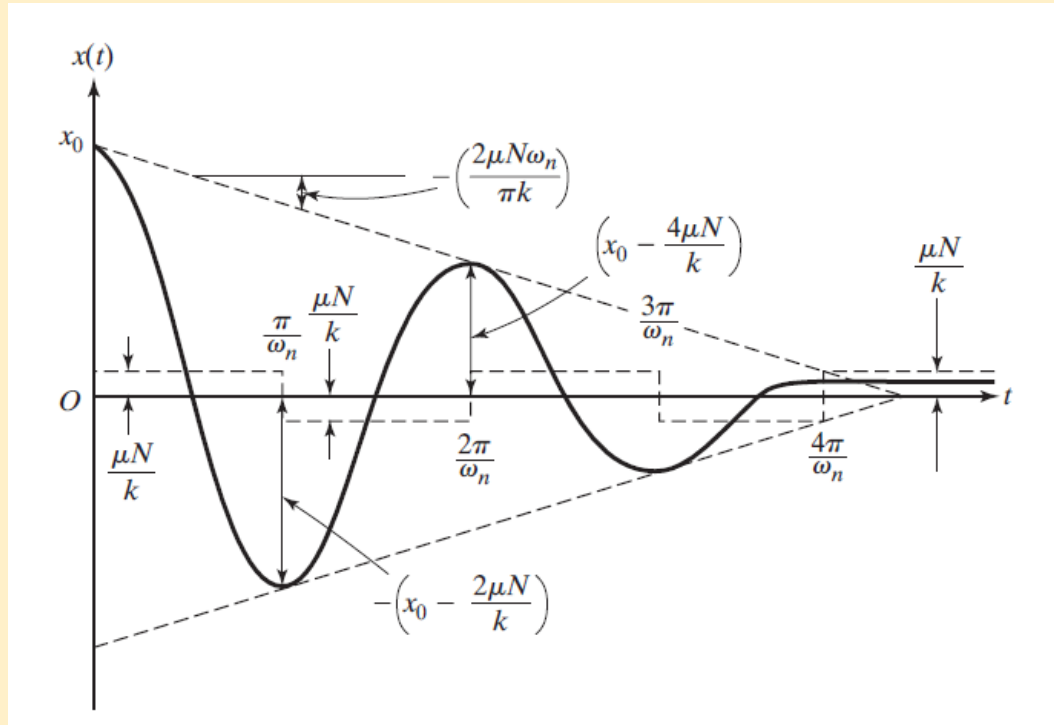
$$\dot{x}(t = 0) = 0$$

$$-A_1 = -x_0 + \frac{3\mu N}{k}, \quad A_2 = 0$$

$$x(t) = \left(x_0 - \frac{3\mu N}{k}\right) \cos \omega_n t - \frac{\mu N}{k}$$

$$\pi / \omega_n \leq t \leq 2\pi / \omega_n$$

VIBRATION



Single DoF free vibration system

The motion stops when $x_n \leq \mu N / k$.

$$x_0 - r \frac{2\mu N}{k} \leq \frac{\mu N}{k}$$

$$r \geq \left\{ \frac{x_0 - \frac{\mu N}{k}}{\frac{2\mu N}{k}} \right\}$$



Single DoF free vibration system

Note the following characteristics of a system with Coulomb damping:

1. The equation of motion is nonlinear with Coulomb damping, while it is linear with viscous damping.
2. The natural frequency of the system is unaltered with the addition of Coulomb damping, while it is reduced with the addition of viscous damping.
3. The motion is periodic with Coulomb damping, while it can be nonperiodic in a viscously damped (overdamped) system.
4. The system comes to rest after some time with Coulomb damping, whereas the motion theoretically continues forever (perhaps with an infinitesimally small amplitude) with viscous and hysteresis damping.
5. The amplitude reduces linearly with Coulomb damping, whereas it reduces exponentially with viscous damping.
6. In each successive cycle, the amplitude of motion is reduced by the amount $4\mu N/k$, so the amplitudes at the end of any two consecutive cycles are related:

$$X_m = X_{m-1} - \frac{4\mu N}{k}$$

